Practice Problems on Integrals
Solutions

1. Evaluate the following integrals:

   (a) $\int_0^1 (x^3 + 2x^5 + 3x^{10})dx$
   Solution: $(1/4) + 2(1/6) + 3(1/11)$

   (b) $\int_0^\infty (1+x)^{-5}dx$
   Solution: Change variables $y = 1 + x$: $\int_1^\infty y^{-5}dy = 1/4$

   (c) $\int_0^\infty x(1+x)^{-5}dx$
   Solution: Change variables $y = 1+x$: $\int_1^\infty (y-1)y^{-5}dy = \int_1^\infty y^{-4}dy - \int_1^\infty y^{-5}dy = (1/3) - (1/4) = 1/12$

   (d) $\int_1^\infty e^{-3x}dx$
   Solution: $(1/3)e^{-3}$

   (e) $\int_1^\infty xe^{-3x}dx$
   Solution: $(4/9)e^{-3}$ (use integration by parts)

   (f) $\int_{-\infty}^\infty |x|e^{-x^2/2}dx$
   Solution: By symmetry, this is $2\int_0^\infty xe^{-x^2/2}dx$. Substituting $u = x^2$, $du = 2xdx$, this becomes $\int_0^\infty e^{-u/2}du = 2$

2. Given that X has density (p.d.f.)

$$f(x) = \begin{cases} 1 - |x| & \text{for } -1 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

   evaluate:

   (a) $P(X \ge 1/2)$
   Solution: $P(X \ge 1/2) = \int_{1/2}^\infty f(x)dx = \int_{1/2}^1 (1-x)dx = 1/8$. (Alternatively, determine the answer geometrically, as the area under the graph of $f(x)$ from 1/2 to 1)

   (b) $P(X \ge -1/2)$
   Solution: $P(X \ge -1/2) = \int_{-1/2}^\infty f(x)dx = \int_{-1/2}^0 (1+x)dx + \int_0^1 (1-x)dx = 7/8$. (Again, this can also be obtained geometrically, via area considerations.)

   (c) $E(X)$
   Solution: $E(X) = \int_{-\infty}^\infty xf(x)dx = \int_{-1}^0 x(1+x)dx + \int_0^1 x(1-x)dx = 0$.

   (d) $E(X^2)$
   Solution: $E(X^2) = \int_{-1}^1 x^2 f(x)dx = \int_{-1}^0 x^2(1+x)dx + \int_0^1 x^2(1-x)dx = 1/6$.

(e) $F(x)$ (the c.d.f.)

Solution: First, note that for $x < -1$, $F(x) = 0$, and for $x > 1$, $F(x) = 1$, so it remains to consider the range $-1 \le x \le 1$. In this range, $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-1}^{x}(1-|t|)dt$. Because of the absolute value sign in $f(t) = 1 - |t|$, we need to consider separately the cases when $-1 \le x < 0$ and $0 \le x \le 1$, and split the integral at 0 in the latter case. For $-1 \le x < 0$,

$$F(x) = \int_{-1}^{x}(1+t)dt = \left[t + \frac{t^2}{2}\right]_{t=-1}^{t=x} = \left(x + \frac{x^2}{2}\right) - \left((-1) + \frac{(-1)^2}{2}\right) = x + \frac{x^2}{2} + \frac{1}{2}.$$

In particular, $F(-1) = 0$, $F(0) = 1/2$, as expected. For $0 \le x \le 1$,

$$F(x) = \int_{-1}^{0}(1+t)dt + \int_{0}^{x}(1-t)dt = \frac{1}{2} + \left[t - \frac{t^2}{2}\right]_{t=0}^{t=x} = \frac{1}{2} + x - \frac{x^2}{2}.$$

Altogether, $F(x)$ is given by

$$F(x) = \begin{cases} 0 & \text{for } x < -1, \\ x + \frac{x^2}{2} + \frac{1}{2}. & \text{for } -1 \le x \le 0, \\ x - \frac{x^2}{2} + \frac{1}{2}. & \text{for } 0 \le x \le 1, \\ & \text{for } x > 1. \end{cases}$$

3. Let X be exponentially distributed with mean 2. Determine:

(a) $P(X \ge 5)$.

Solution: We have $P(X \ge 5) = e^{-5/\theta} = e^{-5/2}$, by the tail formula for an exponential distribution.

(b) $P(2 \le X \le 5)$.

Solution: We have $F(x) = 1 - e^{-x/2}$ for $x \ge 0$, so $P(2 \le X \le 5) = F(5) - F(2) = (1 - e^{-5/2}) - (1 - e^{-2/2}) = e^{-1} - e^{-5/2}$.

(c) $P(2 < X < 5)$.

Solution: Since X has a continuous distribution, this is the same as $P(2 \le X \le 5)$ computed above.

(d) $P(X \ge 5|X \ge 2)$.

Solution: By the definition of conditional probabilities,

$$\begin{aligned} P(X \ge 5|X \ge 2) &= \frac{P(X \ge 5 \text{ and } X \ge 2)}{P(X \ge 2)} \\ &= \frac{P(X \ge 5)}{P(X \ge 2)} = \frac{e^{-5/2}}{e^{-2/2}} = e^{-3/2}. \end{aligned}$$

(e) $P(X \le 5|X \ge 2)$.

Solution: By the same argument,

$$\begin{aligned} P(X \le 5|X \ge 2) &= \frac{P(X \le 5 \text{ and } X \ge 2)}{P(X \ge 2)} \\ &= \frac{P(2 \le X \le 5)}{P(X \ge 2)} = \frac{e^{-1} - e^{-5/2}}{e^{-2/2}} = 1 - e^{-3/2}. \end{aligned}$$

(Alternatively, one can derive this from the previous part, using the complement rule for conditional probabilities: $P(X \le 5|X \ge 2) = 1 - P(X \ge 5|X \ge 2)$)

4. Suppose X has exponential distribution with median 3. Determine:

   (a) $E(X)$.

   Solution: We are given that $0.5 = F(3) = 1 - e^{-3/\theta}$. Solving for θ gives $\theta = -3/\ln 0.5 = 3/\ln 2$. Hence $E(X) = \theta = 3/\ln 2 = 4.32$.

   (b) The 75-th percentile of the distribution of X.

   Solution: We have $F(x) = 1 - e^{-x/\theta} = 1 - e^{-x(\ln 2)/3}$. To get the 75-th percentile, we set $F(x) = 0.75$ and solve for x: $1 - e^{-x(\ln 2)/3}$=0.75 $x = (-3\ln 0.25/\ln 2) = 3\ln 4/\ln 2) = 3 \cdot 2 = 6$.

5. Let X be exponentially distributed with mean 2, and let Y be defined by

$$Y = \begin{cases} 0 & \text{if } X \le 1, \\ X - 1 & \text{if } X > 1. \end{cases}$$

Find $E(Y)$.

Solution: Integrating by parts, we get

$$\begin{aligned} E(Y) &= \int_1^\infty (x-1)\frac{1}{2}e^{-x/2}dx \\ &= -(x-1)e^{-x/2}\Big|_1^\infty + \int_1^\infty e^{-x/2}dx \\ &= 0 - 2e^{-x/2}\Big|_1^\infty = 2e^{-1/2}. \end{aligned}$$

6. Let X be exponentially distributed with mean 2, and let

$$Y = \begin{cases} X & \text{if } X \le 5, \\ 5 & \text{if } X > 5. \end{cases}$$

Find $E(Y)$.

Solution:

$$\begin{aligned} E(Y) &= \int_0^5 x\frac{1}{2}e^{-x/2}dx + \int_5^\infty 5 \cdot \frac{1}{2}e^{-x/2}dx \\ &= -xe^{-x/2}\Big|_0^5 + \int_0^5 e^{-x/2}dx + \frac{5}{2}\int_5^\infty e^{-x/2}dx \\ &= -5e^{-5/2} + 2(1 - e^{-5/2}) + 5(e^{-5/2}) \\ &= 2(1 - e^{-5/2}) \end{aligned}$$

7. Let X be exponentially distributed with mean 2, and let Y be defined by

$$Y = \begin{cases} X & \text{if } X \le 1, \\ (1/2)(X+1) & \text{if } X > 1. \end{cases}$$

Find $E(Y)$.

Solution:

$$\begin{aligned} E(Y) &= \int_0^1 x\frac{1}{2}e^{-x/2}dx + \int_1^\infty \frac{1}{2}(x+1)\frac{1}{2}e^{-x/2}dx \\ &= -xe^{-x/2}\Big|_0^1 + \int_0^1 e^{-x/2}dx + \left(-\frac{1}{2}(x+1)e^{-x/2}\Big|_1^\infty + \frac{1}{2}\int_1^\infty e^{-x/2}dx\right) \\ &= -e^{-1/2} + 2(1-e^{-1/2}) + \left(2e^{-1/2} + 2e^{-1/2}\right) \\ &= 2(1 - \frac{1}{2}e^{-1/2}) \end{aligned}$$

8. Let X be exponentially distributed with mean 3, and let $Y = \max(X, 2)$. Find $E(Y)$.

Solution: Note that

$$Y = \begin{cases} 2 & \text{if } X \le 2, \\ X & \text{if } X > 2. \end{cases}$$

Thus,

$$\begin{aligned} E(Y) &= \int_0^2 2 \cdot \frac{1}{3}e^{-x/3}dx + \int_2^\infty x \cdot \frac{1}{3}e^{-x/3}dx \\ &= 2(1-e^{-2/3}) - xe^{-x/3}\Big|_2^\infty + \int_2^\infty e^{-x/3}dx \\ &= 2(1-e^{-2/3}) + 2e^{-2/3} + 3e^{-2/3} = 2 + 3e^{-2/3} \end{aligned}$$

9. Assume the amount of damage, X, in an auto accident is exponentially distributed with mean 2. (All figures are thousands of dollars.)

   (a) Suppose first the insurance company covers the actual amount of the loss, up to a maximum of 5. What is the average payoff?

   Solution: Letting Y denote the payoff, we have $Y = \min(X, 100)$, i.e.,

$$Y = \begin{cases} X & \text{if } X \le 5, \\ 5 & \text{if } X > 5. \end{cases}$$

   and we need to compute $E(Y)$. This is the calculation carried out in Problem 6; the result is $E(Y) = 2(1 - e^{-5/2})$.

   (b) Suppose now the insurance company covers the full amount of the loss minus a deductible of 1. What is the average payoff?

Solution: Letting Y denote the payoff, we now have

$$Y = \begin{cases} 0 & \text{if } X \le 1, \\ X - 1 & \text{if } X > 1. \end{cases}$$

We need to compute $E(Y)$. This is the computation carried out in Problem 5; the result is $E(Y) = 2e^{-1/2}$.

(c) Suppose the insurance company covers the full amount of the loss up to 1, and 50% of any loss in excess of 1. What is the average payoff?

Solution: Letting Y denote the payoff, we now have

$$Y = \begin{cases} X & \text{if } X \le 1, \\ 1 + (1/2)(X - 1) = (1/2)(X + 1) & \text{if } X > 1. \end{cases}$$

We need to compute $E(Y)$. By the calculation of Problem 7, we get $E(Y) = 2(1 - \frac{1}{2}e^{-1/2})$.